

# On The Capacity of Surfaces in Manifolds with Nonnegative Scalar Curvature

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## Abstract

Given a surface in an asymptotically flat 3-manifold with nonnegative scalar curvature, we derive an upper bound for the capacity of the surface in terms of the area of the surface and the Willmore functional of the surface. The capacity of a surface is defined to be the energy of the harmonic function which equals 0 on the surface and goes to 1 at  $\infty$ . Even in the special case of  $\mathbb{R}^3$ , this is a new estimate. More generally, equality holds precisely for a spherically symmetric sphere in a spatial Schwarzschild 3-manifold. As applications, we obtain inequalities relating the capacity of the surface to the Hawking mass of the surface and the total mass of the asymptotically flat manifold.

## 1 Introduction

The research in this paper was partly motivated by the following theorem.

**Theorem** ([1]) *Let  $(M^3, g)$  be a complete, asymptotically flat 3-manifold with boundary with nonnegative scalar curvature. Suppose its boundary  $\partial M$  consists of horizons (that is  $\partial M$  has zero mean curvature). Let  $G(x)$  be a function on  $M^3$  which satisfies*

$$\begin{cases} \lim_{x \rightarrow \infty} G &= 1 \\ \Delta G &= 0 \\ G|_{\partial M} &= 0. \end{cases} \quad (1)$$

*Then*

$$m \geq C, \quad (2)$$

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where  $m$  is the total mass of  $(M^3, g)$  and  $C$  is the constant in the asymptotic expansion

$$G(x) = 1 - \frac{C}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{at } \infty. \quad (3)$$

Furthermore, equality holds if and only if the manifold  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold outside its horizon.

This theorem was established to prove the Riemannian Penrose Inequality in [1]. It was later applied in [6] for a generalization of Bunting and Masood's rigidity theorem [3] on static vacuum spacetime with black hole boundary. Considering these applications, it is of interest to know whether a similar theorem holds on an asymptotically flat 3-manifold with a boundary that does not necessarily have zero mean curvature. A corollary to our main result, Theorem 1 stated in a moment, is the following:

**Corollary 1** *Let  $(M^3, g)$  be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature with a connected smooth boundary. Assume  $(M^3, g)$  is diffeomorphic to  $\mathbb{R}^3 \setminus \Omega$ , where  $\Omega$  is a bounded domain. Let  $\mathcal{G}(x)$  be a function on  $(M^3, g)$  which satisfies*

$$\begin{cases} \lim_{x \rightarrow \infty} \mathcal{G} &= 1 \\ \Delta \mathcal{G} &= 0 \\ \mathcal{G}|_{\partial M} &= \sqrt{\frac{1}{16\pi} \int_{\partial M} H^2 d\mu}, \end{cases} \quad (4)$$

where  $H$  is the mean curvature of  $\partial M$  and  $d\mu$  is the induced surface measure. If  $\partial M$  has nonnegative Hawking mass (that is  $\int_{\partial M} H^2 d\mu \leq 16\pi$ ), then

$$m \geq \mathcal{C}, \quad (5)$$

where  $m$  is the total mass of  $(M^3, g)$  and  $\mathcal{C}$  is the constant in the asymptotic expansion

$$\mathcal{G}(x) = 1 - \frac{\mathcal{C}}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{at } \infty. \quad (6)$$

Furthermore, equality holds if and only if the manifold  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold

$$(M_{r_0}, g_m^S) = \left( [r_0, \infty) \times S^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + d\sigma^2 \right),$$

where  $r_0$  is some positive constant satisfying  $r_0 \geq 2m$  and  $d\sigma^2$  is the standard metric on the unit sphere  $S^2 \subset \mathbb{R}^3$ .

Before we state our main result, we first define the capacity of a surface.

**Definition 1** Let  $(M^3, g)$  be a complete, asymptotically flat 3-manifold with a nonempty boundary  $\Sigma$ . The **capacity** of  $\Sigma$  in  $(M^3, g)$ , denoted by  $C_M(\Sigma, g)$ , is defined to be

$$C_M(\Sigma, g) = \inf \left\{ \frac{1}{4\pi} \int_{M^3} |\nabla \phi|^2 dg \right\}, \quad (7)$$

where the infimum is taken over all locally Lipschitz  $\phi(x)$  which go to 1 at  $\infty$  and equal 0 on  $\Sigma$ .

When  $(M^3, g)$  is the complement of a smooth bounded domain  $\Omega$  in the Euclidean space  $(\mathbb{R}^3, g_0)$ ,  $C_M(\partial\Omega, g_0)$  is simply the usual electrostatic capacity of  $\partial\Omega$  [7]. In this case, we write  $C_M(\partial\Omega, g_0)$  as  $C(\partial\Omega)$ . Our main theorem is the following:

**Theorem 1** Let  $(M^3, g)$  be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature with a connected smooth boundary. Assume  $(M^3, g)$  is diffeomorphic to  $\mathbb{R}^3 \setminus \Omega$ , where  $\Omega$  is a bounded domain. Then

$$C_M(\partial M, g) \leq \sqrt{\frac{|\partial M|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\partial M} H^2 d\mu} \right), \quad (8)$$

where  $|\partial M|$  and  $H$  are the area and the mean curvature of  $\partial M$ . Furthermore, equality holds if and only if  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold

$$(M_{r_0}, g_m^S) = \left( [r_0, \infty) \times S^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + d\sigma^2 \right),$$

where  $r_0$  is some positive constant satisfying  $r_0 \geq 2m$  and  $d\sigma^2$  is the standard metric on the unit sphere  $S^2 \subset \mathbb{R}^3$ .

As an immediate corollary, we have a new estimate of the capacity of a surface in  $(\mathbb{R}^3, g_0)$ .

**Corollary 2** Let  $\Omega \subset (\mathbb{R}^3, g_0)$  be a bounded domain with a connected smooth boundary. Then

$$C(\partial\Omega) \leq \sqrt{\frac{|\partial\Omega|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\partial\Omega} H^2 d\mu} \right). \quad (9)$$

Furthermore, equality holds if and only if  $\Omega$  is a round ball.

We note that it is interesting to compare (9) with the classical isoperimetric inequality in  $\mathbb{R}^3$ :

$$\left( \frac{3V}{4\pi} \right)^{\frac{1}{3}} \leq \sqrt{\frac{|\partial\Omega|}{4\pi}} = \sqrt{\frac{|\partial\Omega|}{16\pi}} (1 + 1), \quad (10)$$

where  $V$  is the volume of  $\Omega \subset \mathbb{R}^3$ . It is known, among all domains  $\Omega$  with a fixed amount volume  $V > 0$ ,  $C(\partial\Omega)$  is minimized by a round ball [7], i.e.

$$\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \leq C(\partial\Omega). \quad (11)$$

On the other hand, the Willmore functional  $\int_{\partial\Omega} H^2 d\mu$  satisfies

$$\int_{\partial\Omega} H^2 d\mu \geq 16\pi.$$

Hence, (9) is analogous to the classical isoperimetric inequality (10), but where both sides of the inequality are increased.

We outline the idea of the proofs of Corollary 1 and Theorem 1. For both results, we apply the technique of weak inverse mean curvature flow as developed by Huisken and Ilmanen in [4]. The topological assumptions on  $M^3$  ensures that the Hawking mass of the flowing surface  $\Sigma_t$  is monotone nondecreasing for positive  $t$ . A key step in the proof of Theorem 1 is to use the flow to construct a special test function that gives the estimate (8). Such a construction was first used by Bray and Neves in [2]. It is a convenient feature of Corollary 1 and Theorem 1 that, though they are proved by applying the weak inverse mean curvature flow technique, they hold *without* assuming the boundary surface is *outer minimizing* (see [1] [4]).

This paper is organized as follows. In Section 2, we recall some classical results on capacity of convex surfaces in  $\mathbb{R}^3$  from the work of Pólya and Szegő in [7] to illustrate that the weak inverse mean curvature flow technique fits naturally with the classical method. In Section 3, we apply the theory of weak inverse mean curvature flow to prove a general theorem on  $C_M(\Sigma, g)$ , which includes Theorem 1 as a special case. In Section 4, we relate our estimate on  $C_M(\Sigma, g)$  to estimates of the Hawking mass and the total mass, and prove Corollary 1. In Section 5, we give an application of Corollary 1 to the study of static metrics in general relativity.

## 2 Capacity of convex surfaces in $\mathbb{R}^3$

We first give an account of some classical methods and results from [7] in estimating  $C(\Sigma)$  for convex surfaces  $\Sigma$  in  $\mathbb{R}^3$ .

Let  $\Sigma$  be a closed, connected  $C^2$  surface bounding some domain  $\Omega$  in  $\mathbb{R}^3$ . One basic idea in estimating  $C(\Sigma)$  is to minimize  $\int_{\mathbb{R}^3 \setminus \Omega} |\nabla v|^2 dg_0$  over functions  $v$  which have *given level surfaces*  $\{\Sigma_t\}$ . These level surfaces form a one-parameter family. Therefore, after the selection of  $\{\Sigma_t\}$ ,  $v$  becomes a function of one variable and the infimum of  $\int_{\mathbb{R}^3 \setminus \Omega} |\nabla v|^2 dg_0$  over all such  $v$

can be easily evaluated. Precisely, we fix a function  $\psi$ , defined on  $\mathbb{R}^3 \setminus \Omega$ , which satisfies

$$\psi \geq 0, \quad \psi|_{\Sigma} = 0, \quad \lim_{x \rightarrow \infty} \psi = \infty.$$

Let  $\{\Sigma_t = \psi^{-1}(t) \mid 0 \leq t < \infty\}$  be the family of level surfaces of  $\psi$ . For any other function  $v$  having the same level surfaces  $\{\Sigma_t\}$ ,  $v$  must have the form  $v(x) = f(\psi(x))$  for some single variable function  $f(t)$ , which satisfies  $f(0) = 0$  and  $f(\infty) = 1$ . By the co-area formula, we have

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \Omega} |\nabla v|^2 dg_0 &= \int_0^\infty \left( \int_{\Sigma_t} f'(t)^2 |\nabla \psi| d\mu \right) dt \\ &= \int_0^\infty f'(t)^2 \left( \int_{\Sigma_t} |\nabla \psi| d\mu \right) dt. \end{aligned} \quad (12)$$

Define

$$T(t) = \frac{1}{4\pi} \int_{\Sigma_t} |\nabla \psi| d\mu, \quad (13)$$

which is determined solely by the level surfaces  $\{\Sigma_t\}$ . Then

$$C(\Sigma) \leq \int_0^\infty f'(t)^2 T(t) dt. \quad (14)$$

Applying the fundamental theorem of calculus and the Hölder inequality,

$$\begin{aligned} 1 &= \left( \int_0^\infty f'(t) dt \right)^2 = \left( \int_0^\infty f'(t) T(t)^{\frac{1}{2}} T(t)^{-\frac{1}{2}} dt \right)^2 \\ &\leq \left( \int_0^\infty f'(t)^2 T(t) dt \right) \left( \int_0^\infty T(t)^{-1} dt \right). \end{aligned} \quad (15)$$

Thus

$$\left( \int_0^\infty T(t)^{-1} dt \right)^{-1} \leq \int_0^\infty f'(t)^2 T(t) dt \quad (16)$$

for all  $f(t)$  with equality if and only if

$$f(t) = \Lambda \int_0^t \frac{1}{T(s)} ds, \quad (17)$$

where  $\Lambda = \left( \int_0^\infty T(t)^{-1} dt \right)^{-1}$ . Choosing such a  $f(t)$ , we show that

$$C(\Sigma) \leq \left( \int_0^\infty T(t)^{-1} dt \right)^{-1}. \quad (18)$$

Now suppose  $\Sigma$  is a *convex* surface in  $\mathbb{R}^3$ . A natural choice for  $\{\Sigma_t\}$  is the family of level surfaces of the distance function to  $\Sigma$ . We let  $\psi(x) = \text{dist}(x, \Sigma)$  and define

$$\Sigma_t = \{x \mid \text{dist}(x, \Sigma) = t\}. \quad (19)$$

Then  $|\nabla\psi| = 1$  everywhere and  $T(t) = \frac{|\Sigma_t|}{4\pi}$ , where  $|\Sigma_t|$  is the area of  $\Sigma_t$ , given by

$$|\Sigma_t| = |\Sigma| + \left( \int_{\Sigma} H d\mu \right) t + 4\pi t^2. \quad (20)$$

This leads to the following theorem of Szegő.

**Theorem** ([7]) *If  $\Sigma$  is a convex surface in  $(\mathbb{R}^3, g_0)$ , then*

$$C(\Sigma) \leq \frac{M}{4\pi} \frac{2\epsilon}{\log \frac{1+\epsilon}{1-\epsilon}}, \quad (21)$$

where

$$M = \frac{1}{2} \int_{\Sigma} H d\mu \text{ and } \epsilon^2 = 1 - \frac{4\pi|\Sigma|}{M^2}. \quad (22)$$

### 3 Capacity of surfaces in asymptotically flat manifolds

Obviously, most of the calculations in Section 2 work on any non-compact Riemannian manifold. The key step is to make a good choice of  $\{\Sigma_t\}$  so that the corresponding  $T(t)$  can be efficiently estimated. In this section, we consider *asymptotically flat 3-manifolds*, on which the theory of weak inverse mean curvature flow developed by Huisken and Ilmanen [4] gives a nearly canonical foliation.

**Definition 2** *A Riemannian 3-manifold  $(M^3, g)$  is said to be **asymptotically flat** if there is a compact set  $K \subset M$  such that  $M \setminus K$  is diffeomorphic to  $\mathbb{R}^3$  minus a compact set and in the coordinate chart defined by this diffeomorphism,*

$$g = \sum_{i,j} g_{ij}(x) dx^i dx^j,$$

where

$$g_{ij} = \delta_{ij} + O(|x|^{-1}), \quad \partial_k g_{ij} = O(|x|^{-2}) \quad \partial_l \partial_k g_{ij} = O(|x|^{-3}).$$

**Theorem** ([4]) *Let  $(M^3, g)$  be a complete, connected asymptotically flat 3-manifold with a  $C^{1,1}$  boundary  $\Sigma$ . There exists a proper, locally Lipschitz function  $\phi \geq 0$  on  $M$ , called the solution to the weak inverse mean curvature flow with initial condition  $\Sigma$ , which satisfies the following properties:*

1.  $\phi|_{\Sigma} = 0$ ,  $\lim_{x \rightarrow \infty} \phi = \infty$ . For  $t > 0$ ,  $\Sigma_t = \partial\{\phi \geq t\}$  and  $\Sigma'_t = \partial\{\phi > t\}$  define an increasing family of  $C^{1,\alpha}$  surfaces.

2. The surfaces  $\Sigma_t$  ( $\Sigma'_t$ ) minimize (strictly minimize) area among surfaces homologous to  $\Sigma_t$  in the region  $\{\phi \geq t\}$ . The surface  $\Sigma' = \partial\{\phi > 0\}$  strictly minimizes area among surfaces homologous to  $\Sigma$  in  $M$ .
3. For almost all  $t > 0$ , the weak mean curvature of  $\Sigma_t$  is defined and equals  $|\nabla\phi|$ , which is positive for almost all  $x \in \Sigma_t$ .
4. For each  $t > 0$ ,  $|\Sigma_t| = e^t|\Sigma'|$ , and  $|\Sigma_t| = e^t|\Sigma|$  if  $\Sigma$  is outer minimizing (that is  $\Sigma$  minimizes area among all surfaces homologous to  $\Sigma$  in  $M$ ).
5. If  $(M^3, g)$  has nonnegative scalar curvature and  $\chi(\Sigma_t) \leq 2$  for all  $t > 0$ , the Hawking mass

$$m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_t} H^2 d\mu \right)$$

is monotone nondecreasing for  $t > 0$  and  $\lim_{t \rightarrow 0+} m_H(\Sigma_t) \geq m_H(\Sigma')$ . Here  $\chi(S)$  is the Euler characteristic of a surface  $S$ .

We note that when  $\phi$  is a smooth function with non-vanishing gradient, property 3 is just saying that the level surfaces  $\{\Sigma_t\}$  move at a speed equal to the inverse of their mean curvature. We are now ready to prove the main result of this section.

**Theorem 2** *Let  $(M^3, g)$  be a complete, connected asymptotically flat 3-manifold which has a  $C^2$  boundary  $\Sigma$ . Suppose  $(M^3, g)$  has nonnegative scalar curvature and the solution to the weak inverse mean curvature flow with initial condition  $\Sigma$  satisfies  $\chi(\Sigma_t) \leq 2$  for all  $t > 0$ . Then*

$$C_M(\Sigma, g) \leq \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma} H^2 d\mu} \right), \quad (23)$$

where  $|\Sigma|$  and  $H$  denote the area and the mean curvature of  $\Sigma$ . Furthermore, equality holds if and only  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold

$$(M_{r_0}, g_m^S) = \left( [r_0, \infty) \times S^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + d\sigma^2 \right), \quad (24)$$

where  $r_0$  is some positive constant satisfying  $r_0 \geq 2m$  and  $d\sigma^2$  is the standard metric on the unit sphere  $S^2 \subset \mathbb{R}^3$ .

*Proof:* We estimate  $\int_M |\nabla v|^2 dg$  for functions  $v$  that have the same level surfaces  $\{\Sigma_t\}$  as the function  $\phi$ , which is the solution to the weak inverse

mean curvature flow starting at  $\Sigma$ . It follows from the calculation in Section 2 that

$$C_M(\Sigma, g) \leq \inf_f \left\{ \int_0^\infty f'(t)^2 T(t) dt \right\}, \quad (25)$$

where the infimum is taken over all  $f(t)$  satisfying  $f(0) = 0$  and  $f(\infty) = 1$  and

$$T(t) = \frac{1}{4\pi} \int_{\Sigma_t} |\nabla \phi| d\mu. \quad (26)$$

Now, for a.e.  $t > 0$ ,

$$\int_{\Sigma_t} |\nabla \phi| d\mu = \int_{\Sigma_t} H d\mu, \quad (27)$$

where  $H$  is the weak mean curvature of  $\Sigma_t$ . To proceed, we make use of the key property that  $m_H(\Sigma_t)$  is monotone nondecreasing for  $t > 0$  and  $m_H(\Sigma') \leq \lim_{s \rightarrow 0+} m_H(\Sigma_s)$ , which are guaranteed by the assumption that  $(M^3, g)$  has nonnegative scalar curvature and  $\chi(\Sigma_t) \leq 2$ . Hence, for each  $t > 0$ ,

$$m_H(\Sigma') \leq m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_t} H^2 d\mu \right). \quad (28)$$

This, together with the Hölder inequality, implies

$$\frac{1}{16\pi|\Sigma_t|} \left( \int_{\Sigma_t} H d\mu \right)^2 \leq \frac{1}{16\pi} \int_{\Sigma_t} H^2 d\mu \leq 1 - m_H(\Sigma') \sqrt{\frac{16\pi}{|\Sigma_t|}}. \quad (29)$$

Hence,

$$\begin{aligned} 4\pi T(t) = \int_{\Sigma_t} H d\mu &\leq \left[ 16\pi|\Sigma_t| \left( 1 - m_H(\Sigma') \sqrt{\frac{16\pi}{|\Sigma_t|}} \right) \right]^{\frac{1}{2}} \\ &= \left[ 16\pi|\Sigma'| e^t \left( 1 - m_H(\Sigma') \sqrt{\frac{16\pi}{|\Sigma'|}} e^{-\frac{t}{2}} \right) \right]^{\frac{1}{2}}, \end{aligned} \quad (30)$$

by the fact  $|\Sigma_t| = e^t |\Sigma'|$ . Now write  $\bar{m}_0 = m_H(\Sigma')$  and  $\bar{A}_0 = |\Sigma'|$ , it follows from (25) and (30) that

$$4\pi C_M(\Sigma, g) \leq \inf_f \left\{ \int_0^\infty f'(t)^2 F(\bar{A}_0, \bar{m}_0, t) dt \right\}, \quad (31)$$

where  $F(\bar{A}_0, \bar{m}_0, t)$  is an explicit function of  $\bar{A}_0$ ,  $\bar{m}_0$  and  $t$ , given by

$$F(\bar{A}_0, \bar{m}_0, t) = \left[ 16\pi \bar{A}_0 e^t \left( 1 - \bar{m}_0 \sqrt{\frac{16\pi}{\bar{A}_0}} e^{-\frac{t}{2}} \right) \right]^{\frac{1}{2}}. \quad (32)$$



To calculate

$$\inf_f \left\{ \int_0^\infty f'(t)^2 F(\bar{A}_0, \bar{m}_0, t) dt \right\},$$

we consider the 3-dimensional spatial Schwarzschild metric

$$g_{\bar{m}_0}^S = \frac{1}{1 - \frac{2\bar{m}_0}{r}} dr^2 + r^2 d\sigma^2.$$

When  $\bar{m}_0 < 0$ ,  $g_{\bar{m}_0}^S$  is defined on  $(0, \infty) \times S^2$  (the metric has a singularity at  $r = 0$ ). When  $\bar{m}_0 \geq 0$ ,  $g_{\bar{m}_0}^S$  is defined on  $[2\bar{m}_0, \infty) \times S^2$ . In either case,  $g_{\bar{m}_0}^S$  is well defined on  $[\bar{r}_0, \infty) \times S^2$ , where  $\bar{r}_0$  satisfies  $\bar{A}_0 = 4\pi\bar{r}_0^2$ . For convenience, we let  $M_{\bar{r}_0}$  denote  $[\bar{r}_0, \infty) \times S^2$ , then the spatial Schwarzschild manifold  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  has a boundary  $\{r = \bar{r}_0\}$  whose area is  $\bar{A}_0$ . A basic fact about  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  is that the classic inverse mean curvature flow in  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  with initial condition  $\{r = \bar{r}_0\}$  is given by the family of coordinate spheres

$$S_t = \{r = \bar{r}_0 e^{\frac{1}{2}t}\}, \quad (33)$$

which have constant mean curvature (depending on  $t$ ) and constant Hawking mass  $\bar{m}_0$ . Therefore, the corresponding function  $\phi(x)$  on  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  is given by

$$\phi(x) = 2 \log \left( \frac{r}{\bar{r}_0} \right). \quad (34)$$

Next, we consider the harmonic function  $u$  on  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  that equals 0 at  $\{r = \bar{r}_0\}$  and goes to 1 at  $\infty$ . We have two cases:

**Case 1**  $\bar{m}_0 \neq 0$ :

In this case, the function

$$v = \sqrt{1 - \frac{2\bar{m}_0}{r}} = 1 - \frac{\bar{m}_0}{r} + O\left(\frac{1}{r^2}\right) \quad (35)$$

is a non-constant harmonic function on  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$ . Hence,

$$u = \frac{v - v(\bar{r}_0)}{1 - v(\bar{r}_0)}. \quad (36)$$

The capacity of the boundary of  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  is

$$C_{M_{\bar{r}_0}}(\partial M_{\bar{r}_0}, g_{\bar{m}_0}^S) = \frac{1}{4\pi} \int_{M_{\bar{r}_0}} |\nabla u|^2 dg_{\bar{m}_0}^S = \frac{\bar{m}_0}{1 - v(\bar{r}_0)}. \quad (37)$$

Note that  $u$  can be rewritten as

$$u = f_0 \circ \phi, \quad (38)$$

where

$$f_0(t) = \frac{1}{1 - v(\bar{r}_0)} \left[ \sqrt{1 - \frac{2\bar{m}_0}{\bar{r}_0 e^{\frac{t}{2}}}} - v(\bar{r}_0) \right]. \quad (39)$$

It then follows from (37), (38) and the fact that  $S_t$  has constant mean curvature and  $m_H(S_t) = \bar{m}_0$  for all  $t$  that  $f_0$  achieves

$$\inf_f \left\{ \int_0^\infty f'(t)^2 F(\bar{A}_0, \bar{m}_0, t) dt \right\}$$

and the infimum is given by

$$\begin{aligned} \int_{M_{\bar{r}_0}} |\nabla u|^2 &= 4\pi \frac{\bar{m}_0}{1 - v(\bar{r}_0)} \\ &= 4\pi \sqrt{\frac{|\Sigma'|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma'} H^2 d\mu} \right). \end{aligned} \quad (40)$$

Going back to (31), we have

$$C_M(\Sigma, g) \leq \sqrt{\frac{|\Sigma'|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma'} H^2 d\mu} \right). \quad (41)$$

**Case 2**  $\bar{m}_0 = 0$ :

In this case, our model space  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  is the Euclidean space  $(\mathbb{R}^3, g_0)$  minus a round ball of radius  $\bar{r}_0$  centered at the origin. Hence,

$$u = 1 - \frac{\bar{r}_0}{r}. \quad (42)$$

The capacity of the boundary of  $(M_{\bar{r}_0}, g_{\bar{m}_0}^S)$  is

$$C(\partial M_{\bar{r}_0}) = \bar{r}_0. \quad (43)$$

Defining

$$f_0(t) = 1 - e^{-\frac{t}{2}}, \quad (44)$$

we can rewrite  $u$  as

$$u = f_0 \circ \phi. \quad (45)$$

The same argument as in the Case 1 then implies that

$$C_M(\Sigma, g) \leq \bar{r}_0 = \sqrt{\frac{|\Sigma'|}{4\pi}} = \sqrt{\frac{|\Sigma'|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma'} H^2 d\mu} \right), \quad (46)$$

where the last equality holds because  $\bar{m}_0 = 0$ .

Therefore, in both cases, we have proved that

$$C_M(\Sigma, g) \leq \sqrt{\frac{|\Sigma'|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma'} H^2 d\mu} \right). \quad (47)$$

To replace  $\Sigma'$  by  $\Sigma$ , we use the property that  $\Sigma'$  strictly minimizes area among all surfaces homologous to  $\Sigma$ . Since  $\Sigma$  is  $C^2$ ,  $\Sigma'$  is  $C^{1,1}$  and  $C^\infty$  where  $\Sigma'$  does not contact  $\Sigma$ . Moreover, the mean curvature  $H'$  of  $\Sigma'$  satisfies

$$H' = 0 \text{ on } \Sigma' \setminus \Sigma \quad \text{and} \quad H' = H \geq 0 \quad \mathcal{H}^2 a.e. \text{ on } \Sigma' \cap \Sigma. \quad (48)$$

In particular, we have

$$|\Sigma'| \leq |\Sigma| \quad \text{and} \quad \int_{\Sigma'} H'^2 d\mu \leq \int_{\Sigma} H^2 d\mu. \quad (49)$$

Therefore, it follows from (47) and (49) that

$$C_M(\Sigma, g) \leq \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma} H^2 d\mu} \right). \quad (50)$$

To complete the proof of Theorem 2, we must consider the case of equality. Suppose

$$C_M(\Sigma, g) = \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma} H^2 d\mu} \right), \quad (51)$$

it follows from the above proof that

$$|\Sigma'| = |\Sigma|, \quad \int_{\Sigma'} H'^2 d\mu = \int_{\Sigma} H^2 d\mu, \quad (52)$$

$$m_H(\Sigma_t) = m_H(\Sigma'), \quad \forall t > 0, \quad (53)$$

and

$$C_M(\Sigma, g) = \frac{1}{4\pi} \int_M |\nabla(f_0 \circ \phi)|^2 dg, \quad (54)$$

where  $f_0$  is either given by (39) or (44) and  $\phi$  is the solution to the weak inverse mean curvature flow in  $(M^3, g)$  with initial condition  $\Sigma$ . It follows from (52) and (53) that  $\Sigma$  is outer minimizing and the Hawking mass  $m_H(\Sigma_t)$  equals  $m_H(\Sigma)$  for every  $t$ . On the other hand, (54) implies that

$$u_M = f_0 \circ \phi \quad (55)$$

is a smooth harmonic function on  $M$ . As a result, the surfaces  $\Sigma_t, \Sigma$  do not “jump” to  $\Sigma'_t, \Sigma'$  (as defined in [4], meaning  $\Sigma_t = \Sigma'_t, \Sigma = \Sigma'$ ), for otherwise the set  $\{x \in M^3 \mid u_M(x) = f_0(t)\}$  would have non-empty interior for some  $t \geq 0$ , contradicting the maximum principle for harmonic functions. Furthermore, applying the maximum principle to the exterior region of  $\Sigma_t$  in  $(M^3, g)$  and using the fact that  $u_M$  is constant on  $\Sigma_t$  and  $\Sigma_t$  is at least  $C^1$ , we conclude that  $\nabla u_M$  never vanishes. Therefore,

$$\phi = f_0^{-1} \circ u_M \quad (56)$$

is a smooth function on  $M$  with non-vanishing gradient. Hence,  $\{\Sigma_t\}$  evolve smoothly at a speed equal to the inverse of their mean curvature. The fact  $m_H(\Sigma_t) = m_H(\Sigma)$  for all  $t > 0$  then readily implies that  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold

$$(M_{r_0}, g_m^S) = \left( [r_0, \infty) \times S^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + d\sigma^2 \right) \quad (57)$$

with  $r_0 \geq 2m$  (see page 423 in [4] for detail). Theorem 2 is proved.  $\square$

Next, we give a topological condition of  $M^3$  that is sufficient to guarantee the assumption  $\chi(\Sigma_t) \leq 2$  in Theorem 2.

**Proposition 1** *Let  $(M^3, g)$  be a complete, connected asymptotically flat 3-manifold with a nonempty boundary  $\Sigma$ . If  $H_2(M, \Sigma) = 0$  and  $\Sigma$  is connected, then  $\{\Sigma_t\}_{t>0}$  remains connected. In particular,  $\chi(\Sigma_t) \leq 2$  for all  $t$ .*

*Proof:* Under the assumption that  $\Sigma$  is connected, Huisken and Ilmanen proved that the sets  $\{\phi < t\}$  and  $\{\phi > t\}$  are connected [4]. They also showed, for each  $t > 0$ ,  $\Sigma_t$  can be approximated in  $C^1$  by earlier surfaces  $\Sigma_s$ , satisfying  $\Sigma_s = \Sigma'_s$ . Hence, we may assume  $\Sigma_t = \{\phi = t\}$ .

Let  $\Sigma_1$  be one component of  $\Sigma_t$ . Since  $H_2(M, \Sigma) = 0$  and  $\Sigma$  is connected, there is a bounded region  $D$  in  $M$  such that either  $\partial D = \Sigma_1$  or  $\partial D = \Sigma_1 \cup \Sigma$ . As the set  $\{\phi > t\}$  is connected and contains  $\infty$ , we have  $\phi \leq t$  on  $D$ . It then follows that  $\phi < t$  on  $D \setminus \Sigma_1$ . Hence  $D \setminus \Sigma_1$  is a component of the set  $\{\phi < t\}$ . Since  $\{\phi < t\}$  is connected, we must have  $D \setminus \Sigma_1 = \{\phi < t\}$ . Therefore,  $\Sigma_1$  is the only component of  $\Sigma_t$  (and  $\partial D = \Sigma_1 \cup \Sigma$ ). We conclude that  $\Sigma_t$  is connected.  $\square$

Theorem 1 now follows directly from Theorem 2, Proposition 1 and the fact  $H_2(\mathbb{R}^3 \setminus \Omega, \partial\Omega) = 0$ .

## 4 Estimate of the total mass

We prove Corollary 1 in this section. First, we point out that Theorem 2 translates directly into a statement about the capacity of  $\Sigma$  and the Hawking mass of  $\Sigma$ .

**Theorem 3** *Let  $(M^3, g)$  be a complete, connected asymptotically flat 3-manifold with a nonempty boundary  $\Sigma$ . Suppose  $(M^3, g)$  satisfies all the assumptions in Theorem 2. Then*

$$|m_H(\Sigma)| \geq |1 - \alpha| C_M(\Sigma, g), \quad (58)$$

where  $\alpha$  is a constant defined by

$$\alpha = \sqrt{\frac{1}{16\pi} \int_{\Sigma} H^2 d\mu}. \quad (59)$$

Furthermore, in the case  $\alpha \neq 1$ , equality holds if and only if  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold

$$(M_{r_0}, g_m^S) = \left( [r_0, \infty) \times S^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + d\sigma^2 \right), \quad (60)$$

where  $r_0$  is some positive constant satisfying  $r_0 \geq 2m$  and  $d\sigma^2$  is the standard metric on the unit sphere  $S^2 \subset \mathbb{R}^3$ .

*Proof:* It follows directly from Theorem 2 and the definition of the Hawking mass.  $\square$

Next, we recall the definition of the total mass of an asymptotically flat 3-manifold (see [1], [4]).

**Definition 3** *Let  $(M^3, g)$  be an asymptotically flat 3-manifold. The total mass of  $(M^3, g)$  is defined as the limit*

$$m = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \sum_{ij} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j d\sigma, \quad (61)$$

where  $S_r$  is the coordinate sphere  $\{|x| = r\}$ ,  $\nu$  is the coordinate unit normal to  $S_r$  and  $d\sigma$  is the area element of  $S_r$  in the coordinate chart.

An important feature of the weak inverse mean curvature flow, proved by Huisken and Ilmanen in [4], is

**Proposition 2** *Let  $(M^3, g)$  be an asymptotically flat 3-manifold and  $\phi$  be a solution to the weak inverse mean curvature flow, then*

$$m \geq \lim_{t \rightarrow \infty} m_H(\Sigma_t), \quad (62)$$

where  $m$  is the total mass of  $(M^3, g)$  and  $\Sigma_t = \partial\{\phi \geq t\}$ .

The next theorem shows that the total mass  $m$  is bounded from below by the same quantity  $(1 - \alpha)C_M(\Sigma, g)$  as in (58). A convenient feature of the theorem is that it does not require  $\Sigma$  to be outer minimizing.

**Theorem 4** *Let  $(M^3, g)$  be a complete, connected asymptotically flat 3-manifold with a nonempty boundary  $\Sigma$ . Suppose  $(M^3, g)$  satisfies all the assumptions in Theorem 2 and  $\Sigma$  has nonnegative Hawking mass. Let  $\mathcal{G}(x)$  be a function on  $(M^3, g)$  which satisfies*

$$\begin{cases} \lim_{x \rightarrow \infty} \mathcal{G} &= 1 \\ \Delta \mathcal{G} &= 0 \\ \mathcal{G}|_{\Sigma} &= \alpha, \end{cases} \quad (63)$$

where

$$\alpha = \sqrt{\frac{1}{16\pi} \int_{\Sigma} H^2 d\mu}. \quad (64)$$

Then

$$m \geq \mathcal{C}, \quad (65)$$

where  $m$  is the total mass of  $(M^3, g)$  and  $\mathcal{C}$  is the constant in the asymptotic expansion

$$\mathcal{G}(x) = 1 - \frac{\mathcal{C}}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{at } \infty. \quad (66)$$

Furthermore, the equality holds if and only if the manifold  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold

$$(M_{r_0}, g_m^S) = \left( [r_0, \infty) \times S^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + d\sigma^2 \right),$$

where  $r_0$  is some positive constant satisfying  $r_0 \geq 2m$  and  $d\sigma^2$  is the standard metric on the unit sphere  $S^2 \subset \mathbb{R}^3$ .

*Proof:* We only need to consider the case  $m_H(\Sigma) > 0$  (that is  $\alpha < 1$ ), as the case  $m_H(\Sigma) = 0$  is essentially the proof of the Positive Mass Theorem via the inverse mean curvature flow [4].

We use the same notations as in the proof of Theorem 2. Applying Proposition 2 and using the monotonicity of the Hawking mass, we have

$$m \geq \lim_{t \rightarrow \infty} m_H(\Sigma_t) \geq \lim_{t \rightarrow 0+} m_H(\Sigma_t) \geq m_H(\Sigma'). \quad (67)$$

The proof of Theorem 2 shows

$$C_M(\Sigma', g) \leq \sqrt{\frac{|\Sigma'|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma'} H^2 d\mu} \right). \quad (68)$$

Let  $\alpha' = \sqrt{\frac{1}{16\pi} \int_{\Sigma'} H^2 d\mu}$ , by (49) we have

$$\alpha' \leq \alpha. \quad (69)$$

Thus,  $\alpha' < 1$ . Hence, (68) is equivalent to

$$m_H(\Sigma') \geq (1 - \alpha') C_M(\Sigma', g). \quad (70)$$

On the other hand, it follows from the maximum principle that

$$C_M(\Sigma', g) \geq C_M(\Sigma, g) \quad (71)$$

with equality if and only if  $\Sigma = \Sigma'$ . Therefore, it follows from (67), (70), (69) and (71) that

$$m \geq (1 - \alpha) C_M(\Sigma, g) \quad (72)$$

with equality if and only if  $\Sigma = \Sigma'$  and

$$C_M(\Sigma, g) = \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 + \sqrt{\frac{1}{16\pi} \int_{\Sigma} H^2 d\mu} \right). \quad (73)$$

By Theorem 2, (73) holds if and only if  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold  $(M_{r_0}, g_m^S)$ . As  $\mathcal{C} = (1 - \alpha) C_M(\Sigma, g)$ , Theorem 4 is proved.  $\square$

Corollary 1 follows directly from Theorem 4, Proposition 1 and the fact  $H_2(\mathbb{R}^3 \setminus \Omega, \partial\Omega) = 0$ .

## 5 Application to static metrics

In this section, we give a simple application of Corollary 1 to the study of static metrics in general relativity.

We recall that a 3-dimensional asymptotically flat manifold  $(M^3, g)$  is called **static** [5] if there is a positive function  $N$ , called the static potential function of  $(M^3, g)$ , satisfying  $N \rightarrow 1$  at  $\infty$  and

$$\begin{cases} N Ric(g) &= D^2 N \\ \Delta N &= 0, \end{cases} \quad (74)$$

where  $D^2 N$  is the Hessian of  $N$  and  $Ric(g)$  is the Ricci curvature of  $g$ . It can be easily checked that  $(M^3, g)$  and  $N$  satisfy (74) if and only if the asymptotically flat spacetime metric  $\bar{g} = -N^2 dt^2 + g$  solves the Vacuum Einstein Equation on  $M \times \mathbb{R}$ . In particular, (74) implies that  $(M^3, g)$  has zero scalar curvature.

A fundamental result in the study of static, asymptotically flat manifolds with boundary is the following black hole uniqueness theorem, proved by Bunting and Masood-ul-Alam [3].

**Theorem** ([3]) *Let  $(M^3, g)$  be a static, asymptotically flat manifold with a nonempty smooth boundary. Let  $N$  be the static potential function of  $(M^3, g)$ . If  $N$  satisfies*

$$N|_{\partial M} = 0, \quad (75)$$

*then  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold outside its horizon.*

The following theorem is a direct application of Corollary 1 and the maximum principle.

**Theorem 5** *Let  $(M^3, g)$  be a static, asymptotically flat manifold with a connected smooth boundary  $\Sigma$ . Assume that  $(M^3, g)$  is diffeomorphic to  $\mathbb{R}^3 \setminus \Omega$ , where  $\Omega$  is a bounded domain. Let  $N$  be the static potential function of  $(M^3, g)$ . If  $\Sigma$  has nonnegative Hawking mass, then*

$$\min_{\Sigma} N^2 \leq \frac{1}{16\pi} \int_{\Sigma} H^2 d\mu. \quad (76)$$

*Furthermore, equality holds if and only if  $(M^3, g)$  is isometric to a spatial Schwarzschild manifold*

$$(M_{r_0}, g_m^S) = \left( [r_0, \infty) \times S^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + d\sigma^2 \right),$$

*where  $r_0$  is some positive constant satisfying  $r_0 \geq 2m$  and  $d\sigma^2$  is the standard metric on the unit sphere  $S^2 \subset \mathbb{R}^3$ .*



**Remark:** If  $N|_{\Sigma} = 0$ , the static metric system (74) implies that  $\Sigma$  is totally geodesic [3]. Hence the equality in (76) holds automatically. Thus, Theorem 5 can be viewed as a partial generalization of Bunting and Masood-ul-Alam's theorem.

*Proof:* Let  $\alpha = \sqrt{\frac{1}{16\pi} \int_{\Sigma} H^2 d\mu}$  and let  $\mathcal{G}$  be the function on  $M$  defined by

$$\begin{cases} \lim_{x \rightarrow \infty} \mathcal{G} &= 1 \\ \Delta \mathcal{G} &= 0 \\ \mathcal{G}|_{\partial M} &= \alpha. \end{cases} \quad (77)$$

Consider the asymptotic expansions of  $\mathcal{G}$  and  $N$ ,

$$\begin{aligned} \mathcal{G} &= 1 - \frac{\mathcal{C}}{|x|} + O\left(\frac{1}{|x|^2}\right), \text{ as } x \rightarrow \infty \\ N &= 1 - \frac{A}{|x|} + O\left(\frac{1}{|x|^2}\right), \text{ as } x \rightarrow \infty. \end{aligned}$$

Suppose  $\min_{\Sigma} N \geq \alpha$ , the strong maximum principle then implies that

$$A \leq \mathcal{C} \quad (78)$$

with equality if and only if  $N = \mathcal{G}$ . By analyzing the static metric system (74), Bunting and Masood-ul-Alm in [3] showed that

$$A = m, \quad (79)$$

where  $m$  is the total mass of  $(M^3, g)$ . On the other hand, as  $(M^3, g)$  has zero scalar curvature and  $m_H(\Sigma) \geq 0$ , Corollary 1 shows that

$$m \geq \mathcal{C}. \quad (80)$$

Therefore, it follows from (78), (79) and (80) that

$$m = \mathcal{C}$$

and  $N = \mathcal{G}$ . In particular,  $(M^3, g)$  is isometric to  $(M_{r_0}, g_m^S)$  by the rigidity part in Corollary 1. Theorem 5 is proved.  $\square$

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